

Entanglement conditions for integrated-optics multi-port quantum interferometry experiments

Junghye Ryu,^{1,2} Marcin Marciniak,² Marcin Wiesniak,^{2,3} and Marek Żukowski²

¹*Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore, Singapore*

²*Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland*

³*Institute of Informatics, University of Gdańsk, 80-952 Gdańsk, Poland*

Integrated optics allows one to perform interferometric experiments based upon multi-port beam-splitter. To observe entanglement effects one can use multi-mode parametric down-conversion emissions. When the structure of the Hamiltonian governing the emissions has (infinitely) many equivalent Schmidt decompositions into modes (beams), one can have perfect EPR-like correlations of numbers of photons emitted into “conjugate modes” which can be monitored at spatially separated detection stations. We provide series of entanglement conditions for all prime numbers of modes, and show their violations by bright multi-mode squeezed vacuum states. One family of such conditions is given in terms of the usual intensity-related variables. Moreover, we show that an alternative series of conditions expressed in terms averages of observed rates, which is a generalization of the ones given in arXiv:1508.02368, is a much better entanglement indicator. Thus the rates seem to emerge as a powerful concept in quantum optics. Generalizations of the approach are expected.

I. INTRODUCTION

In this work we shall study entanglement conditions for bright multi-mode quantum optical fields of undefined intensities (essentially, photon numbers).

The experiments which we study involve pairs of spatially separated multi-port beam-splitter interferometers, of the kind studied in [1]. Multi-port techniques were tested by Walker [2]. Ideas concerning their use to observe effects related with higher dimensional entanglement one can find in [3]. The fact that the multi-port interferometers can perform any unitary transformations of finite dimensional single photon states was shown by Reck *et al.* [4]. In 2000, it was shown that two-photon higher dimensional entanglement leads to stronger violation of local realism than two-qubit one (numerical results of [5], confirmed analytically in [6] and [7]). Also, two-particle higher dimensional entanglement has specific traits which are not present in qubit systems [8].

The work of Reck *et al.* provides a blueprint for any finite dimensional unitary transformations of single photon states. The transformations are produced with the use of passive optical devices: beam splitters and phase shifters. Such multi-port devices give optical modes coupling which also leads to a unitary relation between the input and output modes. If single photon state is of a photon which is in (mode) beam i is denoted by $|\phi_i\rangle$, and the modes are fully distinguishable (i.e., the states form an orthonormal basis), and the unitary transformation describing the action of the multi-port is $|\phi_k^{out}\rangle = \sum_i U_{ki}|\phi_i\rangle$, then the photon creation operator describing the modes transforms according to

$$a_k^{\dagger out} = \sum_i U_{ki} a_i^{\dagger}. \quad (1)$$

Note that the basis property of the considered single photon state implies the following commutation relations $[a_i, a_j^{\dagger}] = \delta_{ij}$, and $[a_i, a_j] = 0$. Identical relations also hold for the ‘out’ operators. Such devices in principle allow to experimentally/operationally test basic single and two (three, ..., etc.) particles quantum physics and quantum information processes via interference experiments, see e.g., [9]. Here we want to study the extensions of such experiments to the second quantized optical fields, having in mind especially states of light with undefined photon numbers. We investigate non-separability criteria. As an exemplary family of the entangled quantum optical states we shall take the (multi-mode) bright squeezed vacuum. Such states have in theory perfect EPR correlations. We shall show this can be extended to the multi-mode cases. Using this property one can derive entanglement conditions, as separable states cannot have EPR correlations. Such conditions were given for the case of four-mode squeezed vacuum in [10], see also [11]. They are in the form on conditions for correlations of standard polarization measurements, that is for Stokes parameters. However, one can reach more sensitive conditions by replacing the Stokes parameters by ensemble averaged polarization measurements (essentially, in each run we measure the ratios of intensities at the exits of polarization analyzers and then take their averages, see below) [12]. In the case of polarization correlation measurements such an approach has an advantage. The older approach based on the Stokes parameters is giving correlation functions which are averages effectively weighted by (fluctuating) intensities. The approach of [12] allows straightforward quantum mechanical averages of polarization correlation readouts over the ensemble of equivalently prepared runs. The aim of this work is to extend such an approach to number of local modes higher than two, and to study basic consequences of such an approach.

With the ongoing improvements in parametric down-conversion techniques, the birth of integrated optics, and laser imprinting methods to build such devices, the multi-port interferometry experiments, such as those suggested in [1], are becoming feasible. As a matter of fact, important tests of exactly such configurations were recently done, see Schaeff *et al.* [13]. The schemes discussed here are involving parametric down-conversion for higher pump powers, in the case of which we do not have essentially emissions of pairs of correlated photons, but superpositions of multi-pair emissions. Therefore new phenomena need to be studied. At least one should check to what extent the features of two-photon correlations are still present in the case of stronger fields.

We shall consider sets of local interferometers, which are effectively unbiased multi-port beam-splitters [1], which in the case of single photons allow transformations to full sets of mutually unbiased unbiased state bases. Because of the known problems (e.g., $d = 6$) concerning sets of unbiased bases we shall consider only situations in which the number of local modes, d , is a prime. Even for d which is a power of a prime, the algebraic proofs are quite cumbersome, and such cases will be studied in forthcoming papers.

As a by-product of our considerations we shall also get complementarity relations for multi-port interferometry of general quantum optical fields. We shall see that many of the traits of complementarity relations for single particle multi-port measurements still hold for fields of undefined photon numbers. If one replaces intensities at the outputs of such devices by rates (ratios between the observed intensity at the given output divided by the total intensity), such relations are quite elegant.

The other implication of our results is that they undermine the usual paradigm that the quantum coherence properties of optical fields can be revealed by intensity correlation functions, see *any* textbook of Quantum Optics.

The results presented here show that, in the case in which one can use correlations between rates, instead of the usual intensity correlations, one often gains in the visibility of non-classical phenomena. This finding will be additionally supported in forthcoming publications.

II. EPR CORRELATIONS: SOURCES, STATES AND MEASUREMENTS

In Refs. [12, 14], a new Stokes operator to characterize the polarization of quantum light was introduced. The redefined Stokes operators allow one to derive new kinds of Bell inequalities which show strong violations for the four-mode bright squeezed vacuum (BSV) state. One can also derive new entanglement witnesses/indicators which detect entanglement for the states and situations in which the traditional approach, like in [10], fails. The results of [12] are for the measurements of three mutually unbiased, complementary polarizations: e.g., horizontal/vertical, diagonal/anti-diagonal and circular right/left handed. Here, we shall derive generalizations of such entanglement conditions for the multi-mode cases.

The construction of entanglement indicators of [10] and [12] takes as its starting point the fact that for the four-mode BSV one can observe perfect EPR correlations (in many pairs of polarization measurements bases), and that separable states do not have this property. They can be only classically correlated. The optimal choice is to use three mutually unbiased, complementary polarization measurement settings.

Here, as our starting point we take multi-mode emissions in the down-conversion process [1, 9]. The emissions from the parametric down-conversion source are directionally correlated due to the phase matching conditions. In the type-I parametric down-conversion, the pairs of photons of the same frequency are emitted into a cone, in such a way that one can register coincidences into pairs of directions along the cone which lay in the same plane as the pump field, for details see [9]. One can select in principle an arbitrary number of such pairs of such directions, and collect their radiations.

The interaction Hamiltonian which describes such an arrangement has the following form:

$$H = i\gamma \sum_{i=0}^{d-1} a_i^\dagger b_i^\dagger + h.c., \quad (2)$$

where a_i^\dagger and b_i^\dagger are the creation operators of i th signal-idler mode pair, and γ is a coupling constant proportional to the pumping power. The modes a_i are directed (via optical fibers, etc.) to a detection station ‘Alice’, while modes b_i to ‘Bob’. Notice that the Hamiltonian can be put in the followings form: $i\gamma \sum_{i,j=0}^{d-1} \delta_{ij} a_i^\dagger b_j^\dagger + h.c.$ If one takes a $d \times d$ unitary matrix U , one has $\delta_{ij} = \sum_k U_{ki}^* U_{kj}$. Further if one defines $a_k^{\dagger out} = \sum_j U_{kj} a_j^\dagger$, and $b_k^{\dagger out} = \sum_i U_{ki}^* b_i^\dagger$, then one can transform it to an equivalent form:

$$H = i\gamma \sum_{k=0}^{d-1} a_k^{\dagger out} b_k^{\dagger out} + h.c. \quad (3)$$

This symmetry of H implies an invariance of the perfect EPR correlations. Such a transformation can be done using a specific pairs of ‘conjugate’ multi-port interferometers, one for Alice one for Bob. In other words, as the squeezed vacuum state resulting from the application of Hamiltonian H on initial vacuum reveals perfect correlations, such correlations also occur after the pair of local conjugate mode transformations.

Notice that one can consider the U transformations matrices of modes which are associated with unitary transformations leading to unbiased bases for a d -dimensional Hilbert space. This is what we shall do. However to avoid technical problems we shall consider only d which are prime numbers (see introduction).

A. Example

Consider $d = 3$. For three pairs of Schmidt modes, the emitted photon pairs are prepared in the following entangled state:

$$|BSV\rangle = \frac{1}{\cosh^2 \Gamma} \sum_{n=0}^{\infty} \sqrt{\frac{(n+1)(n+2)}{2}} \tanh^n \Gamma |\psi^n\rangle, \quad (4)$$

where

$$|\psi^n\rangle = \sqrt{\frac{2}{(n+1)(n+2)}} \sum_{p+q=0}^n |n-p-q\rangle_{a_0} |p\rangle_{a_1} |q\rangle_{a_2} |n-p-q\rangle_{b_0} |p\rangle_{b_1} |q\rangle_{b_2}. \quad (5)$$

Here, the sum is taken over all combinations of nonnegative integers p and q . The parameter Γ describes ‘gain’ and is dependent on γ and the interaction time (basically equal the length of the non-linear crystal divided by the speed of light).

Our measurement devices consist of an unbiased symmetric multi-port beam-splitter [1] and detectors behind the beam splitters. The unbiased multi-port beam-splitter is defined as an d -input and d -output interferometric device, of the property that light entering via only a single port is split to all output ports, $1/d$ of the intensity into each exit. Following the one photon case [1] one can relate with each exit d th complex roots of unity, values $\omega = \exp(2\pi i/d)$, $\{i = 0, 1, \dots, d-1\}$. The three-output case is shown in Fig. 1. However, as we shall see, this is not a necessary technical tool, and the experience of the authors suggests that such an approach is not optimal, in terms of calculations, for $d > 3$. Still, it will be presented here, as it gives a direct generalization of the conditions in Refs. [10] and [12].

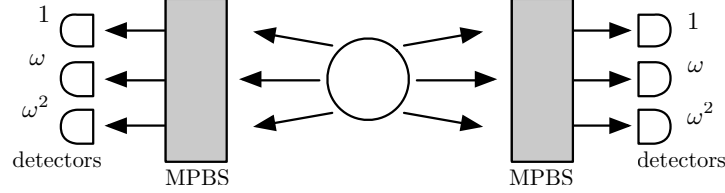


FIG. 1. Schematic diagram of the measurement devices for the three pairs of Schmidt modes. They consist of an three-port beam-splitter (denoted by MPBS) and detectors. By the interaction Hamiltonian (2), the six-mode bright squeezed vacuum state (4) is generated. This entangled state has the perfect EPR correlations between the local conjugate modes.

III. ENTANGLEMENT CONDITION FOR THREE-MODE PAIRS

We shall study here the approach which uses specific value assignment as in Fig. 1 (see [1]). Consider four unitary transformations which lead to the unbiased (complementary) bases in a 3-dimensional Hilbert space (note that when the dimension d of a Hilbert space is an integer power of a prime number, the number of mutually unbiased bases is $d + 1$ [15]). $U(3) = \mathbb{1}$, while the other three, indexed with $m = 0, 1, 2$, have matrix elements which lead to the following transformations of the bases [15]:

$$|\phi_j(m)\rangle = \frac{1}{\sqrt{3}} \sum_{s=0}^2 \omega^{js+ms^2} |s\rangle. \quad (6)$$

With such transformations one can relate a multi-port beam-splitter (interferometer) which couples the input beams the creation operators, a_i^\dagger , with the output ones, $a_j^\dagger(m)$ in the following way:

$$a_j^\dagger(m) = \frac{1}{\sqrt{3}} \sum_{s=0}^2 \omega^{js+ms^2} a_s^\dagger, \quad (7)$$

and we define $a_j^\dagger(m=3) = a_j^\dagger$. As our observables we shall define, for each or the four complementary (local) situations m ,

$$\hat{R}_m = \hat{r}_0(m) + \omega \hat{r}_1(m) + \omega^2 \hat{r}_2(m), \quad (8)$$

where $\hat{r}_j(m) \equiv \Pi \frac{\hat{n}_j(m)}{N} \Pi$, with $\hat{n}_j(m) = a_j^\dagger(m) a_j(m)$, is a photon number operator of j th mode, and $\omega = \exp(2\pi i/3)$. The symbol $\hat{N} = \sum_j \hat{n}_j(m)$ stands for the operator of the total number of photons (it is invariant with respect to unitary mode couplings). Finally, we have $\Pi = \mathbb{1} - |\Omega\rangle\langle\Omega|$, where $|\Omega\rangle$ is a vacuum state. Thanks to this the operator \hat{R}_m acts only in the non-vacuum part of the Fock space of photons. This trick makes the operator $1/\hat{N}$ properly defined.

As the squeezed vacuum state has EPR correlations in measured numbers of photons, one has

$$\sum_{m=0}^3 \left\langle \left| \hat{R}_{m,A} - \hat{R}_{m,B} \right|^2 \right\rangle_{BSV} = 0, \quad (9)$$

where indices A, B denote measurements of Alice and Bob, respectively. We will show that any separable states cannot hold the condition (9). To this end, we use the following two formulas. In Appendix A, we show the following operator relation

$$\sum_{m=0}^3 |\hat{R}_m|^2 = \Pi + \Pi \frac{3}{\hat{N}} \Pi, \quad (10)$$

while in Appendix B we also show that for any separable states $\rho_{sep} = \sum_k p_k \rho_k^A \otimes \rho_k^B$

$$\sum_m \left(\langle \hat{R}_{m,A} \rangle \langle \hat{R}_{m,B}^\dagger \rangle + \langle \hat{R}_{m,A}^\dagger \rangle \langle \hat{R}_{m,B} \rangle \right) \leq 2. \quad (11)$$

Then, one can show that

$$\begin{aligned} & \sum_{m=0}^3 \left\langle \left| \hat{R}_{m,A} - \hat{R}_{m,B} \right|^2 \right\rangle_{sep} \\ &= \sum_m \left(\left\langle \left| \hat{R}_{m,A} \right|^2 \right\rangle + \left\langle \left| \hat{R}_{m,B} \right|^2 \right\rangle - \left(\langle \hat{R}_{m,A} \rangle \langle \hat{R}_{m,B}^\dagger \rangle + \langle \hat{R}_{m,A}^\dagger \rangle \langle \hat{R}_{m,B} \rangle \right) \right) \\ &\geq \sum_k \sum_m p_k \left(\left\langle \left| \hat{R}_{m,A} \right|^2 \right\rangle_A + \left\langle \left| \hat{R}_{m,B} \right|^2 \right\rangle_B - 2 \right) \\ &\geq 3 \left(\left\langle \Pi_A \frac{1}{\hat{N}^A} \Pi_A \right\rangle_A + \left\langle \Pi_B \frac{1}{\hat{N}^B} \Pi_B \right\rangle_B \right), \end{aligned} \quad (12)$$

where $\langle \cdot \rangle_X$ is an expectation for the reduced state ρ^X of a subsystem $X = A, B$. Therefore, for three modes we have a *necessary* condition for a state to be separable, it reads

$$\sum_{m=0}^3 \left\langle \left| \hat{R}_{m,A} - \hat{R}_{m,B} \right|^2 \right\rangle_{sep} \geq 3 \left(\left\langle \Pi_A \frac{1}{\hat{N}^A} \Pi_A \right\rangle_A + \left\langle \Pi_B \frac{1}{\hat{N}^B} \Pi_B \right\rangle_B \right). \quad (13)$$

It is easy to check that analog conditions involving photon numbers rather than rates read:

$$\sum_{m=0}^3 \left\langle \left| \hat{K}_{m,A} - \hat{K}_{m,B} \right|^2 \right\rangle_{sep} \geq 3 \langle \hat{N} \rangle, \quad (14)$$

where $\hat{K}_m = \hat{n}_0(m) + \omega \hat{n}_1(m) + \omega^2 \hat{n}_2(m)$. This is a generalization of the conditions of [10] to $d = 3$. We shall not follow this approach for higher d , as a more general one, without the application of a specific measurement value assignment (i.e., the powers of ω), allows one in a simple way to get results for an arbitrary prime d . We will show the alternative approach in the next section.

In Appendix C we give an analysis of noise resistance of the above entanglement criteria. This is given for our ‘reference’ state, the bright squeezed (six-mode) vacuum. The considered noise is the one for photon losses. We assume that all detectors in the two multi-port experiment of Fig. 1 are of a finite efficiency. The critical efficiencies for the two conditions are very interesting.

For the criterion (14) based on photon numbers rather than the rates, we obtain requirement of $\eta > 1/4$, for *all* values of Γ . Notice that it is a very telling result. It means that, with even ideal detections we cannot get an experiment revealing entanglement of bright squeezed vacuum by splitting the radiation of the source on both sides into four directions (each beam) and then directing each of the branches to four pairs of conjugate complementary interferometers, see Fig. 2 (if this is unclear for the Reader, please note that in the $d = 2$ case of polarized beams, as considered in [12] an equivalent arrangement would be to split the beam leading say to Alice into three of identical intensities, and measuring in each of the three beams, or branches, three fully complementary polarizations, and a similar action on Bob’s side). Such splitting, if one recalls the rule that a passive optical device can be permuted (if this does not lead to a different interferometric setup), acts in each branch in the same way as if the detection efficiency in the branch is $1/4$. Therefore, we see that the intensity based criterion even for perfect efficiency inherits the usual complementarity features of single photon experiments. If one tries to make simultaneously measurements in *all* full complementary situations, the entanglement criterion is worthless. However, this is not so for the criterion based on the rates (13). For a very low gain Γ the critical efficiency is by a whisker below $1/4$ (this is reflecting the fact that the experiment in such a regime is effectively a two-photon one, and standard Bohrian complementarity applies). But for very high Γ one has $\eta_{critical} < 1/4$, as a matter of fact it can be as low as 0.153. Thus we have not

only a better resistance to losses, but additionally, in principle we can detect entanglement of very bright squeezed vacuum by beam-splitting its radiation to each side into four channels and making all the measurements at the same time. The arrangements are not entirely complementary anymore. This hints that with the use of the rates we are probing deeper into the nature of multi-photon light.

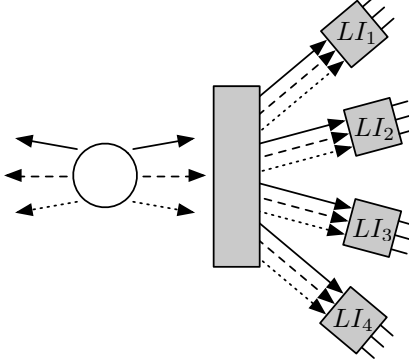


FIG. 2. On Alice's side each beam is split into four (branches). Each such branch is fed to a different local interferometer (LI). Each interferometer is linked with a different mutually unbiased basis (for single photons, see the main text). The same is done on the side of Bob (not shown). Three beams are sent to Alice and three beams to Bob. The figure depicts the arrangement at Alices side (at Bobs it is identical).

Each original beam of Alice is split into four branches (by a cascade of 50-50 beam-splitter, the initial one, and one in the reflected output of the initial one, and one in the transmitted output of the initial one) in such a way that $1/4$ of the intensity always goes to each branch. We have the following four branches: T-T (two transmissions), T-R (transmission first then reflection), R-T (refection then transmission), and R-R (two reflections). All three T-T branches (each from a different original beam) are fed into the interferometer LI_1 which realizes (for single photons) mutually unbiased basis number one (MUB1). All three T-R branches (each from a different original beam) are fed into the interferometer LI_2 which realizes (for single photons) MUB2. All three R-T branches (each from a different original beam) are fed into the interferometer which realizes (for single photons) MUB3. All three R-R branches (each from a different original beam) are fed into the interferometer which realizes (for single photons) MUB4. Thus, for perfect detectors, 100% efficient, counts at conjugated MUB's ($i = 1, 2, 3, 4$) at Alices and Bobs side (the same i on both sides), are identical as if there was just one pair of MUB's (of the same i), like in Fig. 1, and as if the efficiency of detection was $1/4$. Only correlations for (all) pairs conjugated interferometers of Alice and Bob are taken into account. Thus all measurements in all pairs of conjugated bases, considered in the main text, can be done simultaneously, at the price of having a lower effective efficiency. However such an arrangement can detect entanglement only if one use the condition based on the rates. The condition with intensities is useless.

IV. ALTERNATIVE APPROACH FOR ENTANGLEMENT CONDITION

For the squeezed vacuum for p pairs of modes (p is a prime), its perfect correlations give us the following relation

$$\left\langle \sum_{m=0}^p \sum_{j=1}^p [\hat{n}_j^A(m) - \hat{n}_j^B(m)]^2 \right\rangle_{BSV} = 0. \quad (15)$$

The aim now would be to show what is the lower bound of such an expression for any separable states. As the separable states are convex combinations of product ones, the separability condition can be established considering the minimum of the expression for the product state $\rho_A \otimes \rho_B$:

$$\begin{aligned} & \left\langle \sum_{m=0}^p \sum_{j=1}^p [\hat{n}_j^A(m) - \hat{n}_j^B(m)]^2 \right\rangle_{\rho_A \otimes \rho_B} \\ &= \sum_{m,j} \langle \hat{n}_j^A(m)^2 \rangle_{\rho_A} + \sum_{m,j} \langle \hat{n}_j^B(m)^2 \rangle_{\rho_B} - 2 \sum_{m,j} \langle \hat{n}_j^A(m) \rangle_{\rho_A} \langle \hat{n}_j^B(m) \rangle_{\rho_B} \\ &\geq \sum_{m,j} \langle \hat{n}_j^A(m)^2 \rangle_{\rho_A} + \sum_{m,j} \langle \hat{n}_j^B(m)^2 \rangle_{\rho_B} - 2 \left(\sum_{m,j} \langle \hat{n}_j^A(m) \rangle_{\rho_A}^2 \right)^{1/2} \left(\sum_{m,j} \langle \hat{n}_j^B(m) \rangle_{\rho_B}^2 \right)^{1/2} \end{aligned} \quad (16)$$

Let us first consider the last term of (16). In the case of unitary matrices related with generating mutually unbiased bases in Hilbert spaces of prime dimension, the creation operator for j th mode defined by the transformation matrix to the m th mutually unbiased is given by

$$a_j^\dagger(m) = \frac{1}{\sqrt{p}} \sum_{s=0}^{p-1} \omega^{js+ms^2} a_s^\dagger, \quad (17)$$

where the prime p is the number of modes. Using this relation we get the following:

$$\begin{aligned} \sum_{m=0}^p \sum_{j=1}^p \langle \hat{n}_j(m) \rangle^2 &= \sum_{j=1}^p \langle \hat{n}_j \rangle^2 + \frac{1}{p^2} \sum_{m=0}^{p-1} \sum_{j=1}^p \left\langle \sum_{s,t} \omega^{j(s-t)+m(s^2-t^2)} a_s^\dagger a_t \right\rangle^2 \\ &= \sum_j \langle \hat{n}_j \rangle^2 + \frac{1}{p^2} \sum_{m,j} \sum_{s_1,t_1} \sum_{s_2,t_2} \omega^{j(s_1-t_1+s_2-t_2)+m(s_1^2-t_1^2+s_2^2-t_2^2)} \langle a_{s_1}^\dagger a_{t_1} \rangle \langle a_{s_2}^\dagger a_{t_2} \rangle \\ &= \sum_j \langle \hat{n}_j \rangle^2 + \frac{1}{p^2} \sum_{s_1,t_1} \sum_{s_2,t_2} \langle a_{s_1}^\dagger a_{t_1} \rangle \langle a_{s_2}^\dagger a_{t_2} \rangle \sum_j \omega^{j(s_1-t_1+s_2-t_2)} \sum_m \omega^{m(s_1^2-t_1^2+s_2^2-t_2^2)} \\ &= \sum_j \langle \hat{n}_j \rangle^2 + \frac{1}{p^2} \left[p^2 \sum_{s_1,s_2} \langle \hat{n}_{s_1} \rangle \langle \hat{n}_{s_2} \rangle + \sum_{s_1 \neq t_1} \sum_{s_2 \neq t_2} \langle a_{s_1}^\dagger a_{t_1} \rangle \langle a_{s_2}^\dagger a_{t_2} \rangle \sum_j \omega^{j(s_1-t_1+s_2-t_2)} \sum_m \omega^{m(s_1^2-t_1^2+s_2^2-t_2^2)} \right]. \quad (18) \end{aligned}$$

Note, that the summation over m is here from 0 to $p-1$. Consider the case: $s_1 = s_2$, $t_1 = t_2$, $s_1 \neq t_1$, and $s_2 \neq t_2$, then

$$\sum_j \omega^{j(s_1-t_1+s_2-t_2)} = \sum_j \omega^{2j(s_1-t_1)} = 0.$$

Next, if $s_1 = t_2$, $t_1 = s_2$, $s_1 \neq t_1$, and $s_2 \neq t_2$, then

$$\sum_j \omega^{j(s_1-t_1+s_2-t_2)} \sum_m \omega^{m(s_1^2-t_1^2+s_2^2-t_2^2)} = p^2.$$

Finally, for the cases of $s_1 \neq t_1$ and $s_2 \neq t_2$, which are *not* equivalent to the first and second cases, there are two possibilities (*all relations below are modulo p , while “ \rightarrow ” stands for “implies”*):

- $s_1 - t_1 + s_2 - t_2 \neq 0$, then $\sum_j \omega^{j(s_1-t_1+s_2-t_2)} = 0$
- $s_1 - t_1 + s_2 - t_2 = 0 \rightarrow s_2 - t_2 = -(s_1 - t_1)$, then $s_1^2 - t_1^2 + s_2^2 - t_2^2 = (s_1 + t_1)(s_1 - t_1) + (s_2 + t_2)(s_2 - t_2) = (s_1 + t_1)(s_1 - t_1) - (s_2 + t_2)(s_1 - t_1) = (s_1 + t_1 - s_2 - t_2)(s_1 - t_1)$. If $s_1^2 - t_1^2 + s_2^2 - t_2^2 = 0$, then $s_1 + t_1 - s_2 - t_2 = 0$ and $s_1 - t_1 + s_2 - t_2 \rightarrow s_1 - t_2 = 0 \rightarrow s_1 = t_2$.

Thus, Eq. (18) is reduced to

$$\begin{aligned} &\sum_j \langle \hat{n}_j \rangle^2 + \sum_{s_1,s_2} \langle \hat{n}_{s_1} \rangle \langle \hat{n}_{s_2} \rangle + \frac{1}{p^2} \sum_{s_1 \neq t_1} p^2 \langle a_{s_1}^\dagger a_{t_1} \rangle \langle a_{t_1}^\dagger a_{s_1} \rangle \\ &= \sum_j \langle \hat{n}_j \rangle^2 + \langle \hat{N} \rangle^2 + \sum_{s_1 \neq t_1} \langle a_{s_1} \psi | a_{t_1} \psi \rangle \langle a_{t_1} \psi | a_{s_1} \psi \rangle \\ &\leq \sum_j \langle \hat{n}_j \rangle^2 + \langle \hat{N} \rangle^2 + \sum_{s_1 \neq t_1} \|a_{s_1} \psi\|^2 \|a_{t_1} \psi\|^2 \\ &= \sum_j \langle \hat{n}_j \rangle^2 + \langle \hat{N} \rangle^2 + \sum_{s_1 \neq t_1} \langle \hat{n}_{s_1} \rangle \langle \hat{n}_{t_1} \rangle = 2 \langle \hat{N} \rangle^2. \quad (19) \end{aligned}$$

Finally, we have $\sum_{m=0}^p \sum_{j=1}^p \langle \hat{n}_j(m) \rangle^2 \leq 2 \langle \hat{N} \rangle^2$.

In case of the first two terms of (16), we have

$$\begin{aligned}
& \sum_j \hat{n}_j^2 + \frac{1}{p^2} \sum_{m=0}^{p-1} \sum_{j=1}^p \sum_{s_1, t_1, s_2, t_2} \omega^{j(s_1-t_1+s_2-t_2)+m(s_1^2-t_1^2+s_2^2-t_2^2)} a_{s_1}^\dagger a_{t_1} a_{s_2}^\dagger a_{t_2} \\
&= \sum_j \hat{n}_j^2 + \frac{1}{p^2} \sum_{s_1, t_1, s_2, t_2} \left(\sum_j \omega^{j(\dots)} \right) \left(\sum_m \omega^{m(\dots)} \right) a_{s_1}^\dagger a_{t_1} a_{s_2}^\dagger a_{t_2} \\
&= \sum_j \hat{n}_j^2 + \sum_{s_1, s_2} \hat{n}_{s_1} \hat{n}_{s_2} + \sum_{s_1 \neq t_1} a_{s_1}^\dagger a_{t_1} a_{s_1}^\dagger a_{s_1} \\
&= \sum_j \hat{n}_j^2 + \sum_{s_1, s_2} \hat{n}_{s_1} \hat{n}_{s_2} + \sum_{s_1 \neq t_1} a_{s_1}^\dagger a_{s_1} (a_{t_1}^\dagger a_{t_1} + 1) \\
&= \sum_j \hat{n}_j^2 + \sum_{s_1, s_2} \hat{n}_{s_1} \hat{n}_{s_2} + \sum_{s_1 \neq t_1} \hat{n}_{s_1} \hat{n}_{t_1} + (p-1) \sum_{s_1} \hat{n}_{s_1} \\
&= \hat{N}^2 + (p-1)\hat{N} + \hat{N}^2 = 2\hat{N}^2 + (p-1)\hat{N}
\end{aligned} \tag{20}$$

With the help of the results of (19) and (20), finally one derive the separability condition (16) in the form of

$$\begin{aligned}
& \left\langle \sum_{m=0}^p \sum_{j=1}^p [\hat{n}_j^A(m) - \hat{n}_j^B(m)]^2 \right\rangle_{\rho_A \otimes \rho_B} \\
&= 2 \left\langle \hat{N}^A \right\rangle^2 + (p-1) \left\langle \hat{N}^A \right\rangle + 2 \left\langle \hat{N}^B \right\rangle^2 + (p-1) \left\langle \hat{N}^B \right\rangle - 4 \left\langle \hat{N}^A \right\rangle \left\langle \hat{N}^B \right\rangle \\
&\geq (p-1) \left(\left\langle \hat{N}^A \right\rangle + \left\langle \hat{N}^B \right\rangle \right) = (p-1) \left\langle \hat{N} \right\rangle.
\end{aligned} \tag{21}$$

To get an analog conditions for the rates, one has to retrace the above derivation, replacing the number operators by the rates $\hat{r}_j(m) \equiv \Pi \hat{n}_j(m) \frac{1}{N} \Pi$, and using on the way the same tricks as in Appendix B. The condition for the rates reads:

$$\left\langle \sum_{m=0}^p \sum_{j=1}^p [\hat{r}_j^A(m) - \hat{r}_j^B(m)]^2 \right\rangle \geq (p-1) \left(\frac{1}{\left\langle \hat{N}^A \right\rangle} + \frac{1}{\left\langle \hat{N}^B \right\rangle} \right). \tag{22}$$

V. RENORMALIZATION

Following [12], we obtain much more sensitive entanglement conditions (for rates) if in all formulas we use instead of the state in question, ρ , its renormalized version after projecting out the vacuum components

$$\rho' = \Pi_A \otimes \Pi_B \rho \Pi_A \otimes \Pi_B / \|\Pi_A \otimes \Pi_B \rho \Pi_A \otimes \Pi_B\|. \tag{23}$$

Note that for any separable state ρ'_{sep} is also separable, thus all considerations presented here can be repeated for such states.

VI. COMPLEMENTARITY RELATIONS

In Section III, we show a separability condition based on the local operator with the specific measurement assignments (the power of ω). To this end, we use the following relation (for the derivation, see Appendix B)

$$\sum_{m=0}^3 |\langle \hat{R}_m \rangle|^2 \leq 1. \tag{24}$$

Note that this is a complementarity relation for the four possible, mutually exclusive interferometric measurements involving $d = 3$ beams. The interferometers are such that they perform unitary transformations leading to mutually unbiased bases for single photon state. If, say $|\langle \hat{R}_1 \rangle| = 1$, then for all $i \neq 1$ we have $|\langle \hat{R}_i \rangle| = 0$.

Complementarity relations for arbitrary prime d also follow from the relations (19), which read $\sum_m \sum_j \langle \hat{n}_j(m) \rangle^2 \leq 2 \langle \hat{N} \rangle^2$, and their analog for the rates $\hat{r}_j(m)$ reads

$$\sum_{m=0}^p \sum_{j=1}^p \langle \hat{r}_j(m) \rangle^2 \leq 2. \quad (25)$$

Thus, if e.g., $\langle \hat{r}_1(m) \rangle = 1$, then $\sum_m \sum_{j \neq 1} \langle \hat{r}_j(m) \rangle^2 \leq 1$. As a matter of fact, $\langle \hat{r}_1(m) \rangle = 1$ implies that the state in question describes all photons exiting via beam 1 (or exit 1) of the multi-port beam-splitter related with the complementary situation m , and also no vacuum component in the state. This implies that in such a case for all the other complementary situations $m' \neq m$, and each j th exit, one has $\langle \hat{r}_j(m') \rangle = 1/p$. Thus the relation (25) is a form of the usual property of mutually unbiased bases for complementary interferometers and arbitrary optical fields.

VII. SUMMARY AND CLOSING REMARKS

For $2 \times d$ -mode (where d is a prime) quantum optical fields of undefined intensities, we formulate series of entanglement criteria based on the ratios between observed and total intensities. As an example, we consider a d -mode bright squeezed vacuum state. Such optical states have a EPR-like correlations of numbers of photons registered in conjugated modes. With the help of multi-port beam-splitter techniques, we are able to see such correlations. As the critical efficiencies are quite moderate for our entanglement conditions can find application is experiments. Moreover, in case of inefficient detection cases, our approach in terms of rates is able to detect entanglement of quantum optical fields of high intensities.

We have derived both conditions which are more traditional, that is based on correlation of intensities, and conditions, inspired by Ref. [12], which use correlations of rates. The latter ones are capable do detect entanglement where the former fail. All this confirms our conjecture that the correlation functions involving rates rather than intensities can become a useful tool in quantum optics. We expect that one can find benefits by using the rates in various cases, e.g., quantum steering, Hong-Ou-Mandel interference, and etc., see our forthcoming manuscripts.

Our conjecture is that the results can be generalized to all d for which $d+1$ mutually unbiased bases are known to exits.

VIII. ACKNOWLEDGMENTS

This work is a part of EU project BRISQ2. The work is subsidized form funds for science for years 2012-2015 approved for international co-financed project BRISQ2 by Polish Ministry of Science and Higher Education (MNiSW). The team of authors is additionally supported by TEAM project of FNP. We thank prof. M. Chekhova for discussions.

Appendix A: Derivation of Eq. (10)

We here derive Eq. (10) of the main text. To this end, we start from an arbitrary prime d -mode case. Let A^\dagger be a row matrix as $(a_0^\dagger, a_1^\dagger, \dots, a_{d-1}^\dagger)$. Its “column Hermitian conjugate” A involves the annihilation operators. Then, one can put $\hat{R}_m = \Pi A^\dagger M_m A \frac{1}{N} \Pi$, where the $d \times d$ matrix M_m is given by

$$\sum_{j=0}^{d-1} \omega^j |\phi_j(m)\rangle \langle \phi_j(m)|, \quad (A1)$$

which is an analog of \hat{R}_m for a d -dimensional Hilbert space (of single photon states). Of course we put $|\phi_j(m=3)\rangle = |j\rangle$. Such matrices form the unitary generalizations of Pauli operators for d -dimensional Hilbert space. Recall that we here consider d which is prime. If one defines $Z = \sum_{j=0}^{d-1} \omega^j |j\rangle \langle j|$ with and $X = \sum_{j=0}^{d-1} |j\rangle \langle j+1|$ (where necessary, all formulas here are modulo d , with respect to indices) can construct $d+1$ mutually complementary “unitary observables”: $M_d = Z$ and $M_k = \omega^k X Z^{-2k}$ for $k \in \{0, 1, \dots, d-1\}$. The set of $X Z^{-2k}$'s is just a permutation of the set of $X Z^k$'s. One has $Z^0 = \mathbb{1} = (M_k)^d$.

Using the above algebraic relations we can put:

$$\sum_{m=0}^d \left| \hat{R}_m \right|^2 = \Pi \frac{1}{\hat{N}} A^\dagger \left(Z A \Pi A^\dagger Z^\dagger + \sum_{k=0}^{d-1} X Z^k A \Pi A^\dagger (X Z^k)^\dagger \right) A \frac{1}{\hat{N}} \Pi. \quad (\text{A2})$$

Note that the projector Π and \hat{N} commute with each other and with any operators of the form $a_j^\dagger a_i$. We make the following transformations (on the way of which we use the following relations: $\sum_{k=0}^{d-1} \omega^{kl} = d\delta_{0l}$, $[a_i, a_j^\dagger] = \delta_{ij}$ and $a_i^\dagger a_i = \hat{n}_i$):

$$\begin{aligned} \sum_{m=0}^d \left| \hat{R}_m \right|^2 &= \Pi \frac{1}{\hat{N}^2} \sum_{i,j} a_i^\dagger \left[\left(\omega^{i-j} a_i^\dagger a_j^\dagger + \sum_{k=0}^{d-1} \omega^{k(i-j)} a_{i+1}^\dagger a_{j+1}^\dagger \right) \right] a_j \Pi \\ &= \Pi \frac{1}{\hat{N}^2} \left(\sum_{i,j} \omega^{i-j} a_i^\dagger a_i a_j^\dagger a_j + d \sum_i a_i^\dagger a_{i+1} a_{i+1}^\dagger a_i \right) \Pi \\ &= \Pi \frac{1}{\hat{N}^2} \left[\sum_i \left(a_i^\dagger a_i a_i^\dagger a_i + d a_i^\dagger a_{i+1} a_{i+1}^\dagger a_i \right) + \sum_{i \neq j} \omega^{i-j} a_i^\dagger a_i a_j^\dagger a_j \right] \Pi \\ &= \Pi \frac{1}{\hat{N}^2} \left[\sum_i (\hat{n}_i \hat{n}_i + d \hat{n}_i \hat{n}_{i+1}) + d \hat{N} + \sum_{i \neq j} \omega^{i-j} \hat{n}_i \hat{n}_j \right] \Pi, \\ &= \Pi \frac{1}{\hat{N}^2} \left[\sum_i (\hat{n}_i \hat{n}_i + d \hat{n}_i \hat{n}_{i+1}) + d \hat{N} + \sum_{i=0}^{d-1} \left(\sum_{k=1}^{(d-1)/2} (\omega^k + \omega^{-k}) \hat{n}_i \hat{n}_{i+k} \right) \right] \Pi. \end{aligned} \quad (\text{A3})$$

For $d = 3$, Eq. (A3) reads

$$\begin{aligned} &\frac{1}{\hat{N}^2} \Pi \left[\sum_{i=0}^2 (\hat{n}_i \hat{n}_i + 3 \hat{n}_i \hat{n}_{i+1}) + 3 \hat{N} + \sum_{i=0}^2 (\omega + \omega^{-1}) \hat{n}_i \hat{n}_{i+1} \right] \Pi \\ &= \frac{1}{\hat{N}^2} \Pi \left[\sum_i (\hat{n}_i \hat{n}_i + 2 \hat{n}_i \hat{n}_{i+1}) + 3 \hat{N} \right] \Pi \\ &= \frac{1}{\hat{N}^2} \Pi \left(\hat{N}^2 + 3 \hat{N} \right) \Pi = \Pi + \Pi \frac{3}{\hat{N}} \Pi. \end{aligned} \quad (\text{A4})$$

Appendix B: Derivation of Inequality (11)

Here we will derive the Eq. (11) Namely that, the following holds for any separable state:

$$\sum_m \left(\left\langle \hat{R}_{m,A} \right\rangle \left\langle \hat{R}_{m,B}^\dagger \right\rangle + \left\langle \hat{R}_{m,A}^\dagger \right\rangle \left\langle \hat{R}_{m,B} \right\rangle \right) \leq 2. \quad (\text{B1})$$

First we notice that as separable states are of the form of a convex combination $\rho_{sep} = \sum_k p_k \rho_k^A \otimes \rho_k^B$, we can search the minimum of LHS of (11) using

$$\max_{\rho^A, \rho^B} \sum_m \text{Tr} \left[\left(\hat{R}_{m,A} \hat{R}_{m,B}^\dagger + \hat{R}_{m,A}^\dagger \hat{R}_{m,B} \right) \rho^A \otimes \rho^B \right]. \quad (\text{B2})$$

However,

$$\begin{aligned} &\sum_m \text{Tr} \left[\left(\hat{R}_{m,A} \hat{R}_{m,B}^\dagger + \hat{R}_{m,A}^\dagger \hat{R}_{m,B} \right) \rho^A \otimes \rho^B \right] \\ &= \sum_m \left(\left\langle \hat{R}_{m,A} \right\rangle_{\rho^A} \left\langle \hat{R}_{m,B}^\dagger \right\rangle_{\rho^B} + \left\langle \hat{R}_{m,A}^\dagger \right\rangle_{\rho^A} \left\langle \hat{R}_{m,B} \right\rangle_{\rho^B} \right). \end{aligned} \quad (\text{B3})$$

Thus in order to find the minimum we must know the general properties of $\langle \hat{R}_{m,A} \rangle_{\rho^A}$ and $\langle \hat{R}_{m,B} \rangle_{\rho^B}$. Specifically, what will be needed will be the upper bound for

$$\sum_{m=0}^3 \langle \hat{R}_{m,X} \rangle_{\rho^X} \langle \hat{R}_{m,X}^\dagger \rangle_{\rho^X}, \quad (\text{B4})$$

where $X = A, B$, as by Cauchy inequality

$$\max \left(\sum_m \text{Tr} \left[\left(\hat{R}_{m,A} \hat{R}_{m,B}^\dagger + \hat{R}_{m,A}^\dagger \hat{R}_{m,B} \right) \rho^A \otimes \rho^B \right] \right)^2 \leq \max \left(4 \sum_{m=0}^3 |\langle \hat{R}_{m,X} \rangle_{\rho^X}|^2 \right). \quad (\text{B5})$$

To this end we can use the algebraic relations already established in the derivation of (10), as given in the equalities (A3). We take any pure state $|\psi\rangle$, and consider $\langle \hat{R}_m \rangle = \langle \psi | \hat{R}_m | \psi \rangle$. Notice that if one takes the average of $\sum_{m=0}^3 \hat{R}_m \hat{R}_m^\dagger$ and inserts $|\psi'\rangle\langle\psi'| = \Pi|\psi\rangle\langle\psi|\Pi$ between the pairs of conjugated operators, one gets $\sum_{m=0}^3 \langle \hat{R}_m \rangle \langle \hat{R}_m^\dagger \rangle$. This in turn can be analyzed using the first three of the equalities (A3):

$$\begin{aligned} \sum_{m=0}^d \langle \hat{R}_m \rangle \langle \hat{R}_m^\dagger \rangle &= \sum_{i,j} \langle \psi' | \frac{1}{\hat{N}} a_i^\dagger \left[\left(\omega^{i-j} a_i |\psi'\rangle \langle \psi' | a_j^\dagger + \sum_k \omega^{k(i-j)} a_{i+1} |\psi'\rangle \langle \psi' | a_{j+1}^\dagger \right) \right] a_j \frac{1}{\hat{N}} |\psi'\rangle \\ &= \sum_{i,j} \omega^{i-j} \langle \psi' | \frac{1}{\hat{N}} a_i^\dagger a_i |\psi'\rangle \langle \psi' | a_j^\dagger a_j \frac{1}{\hat{N}} |\psi'\rangle + d \sum_i \langle \psi' | \frac{1}{\hat{N}} a_i^\dagger a_{i+1} |\psi'\rangle \langle \psi' | a_{i+1}^\dagger a_i \frac{1}{\hat{N}} |\psi'\rangle \\ &= \sum_i \langle \psi' | \frac{1}{\hat{N}} a_i^\dagger \left(a_i |\psi'\rangle \langle \psi' | a_i^\dagger + d a_{i+1} |\psi'\rangle \langle \psi' | a_{i+1}^\dagger \right) a_i \frac{1}{\hat{N}} |\psi'\rangle + \sum_{i \neq j} \omega^{i-j} \langle \psi' | \frac{1}{\hat{N}} a_i^\dagger a_i |\psi'\rangle \langle \psi' | a_j^\dagger a_j \frac{1}{\hat{N}} |\psi'\rangle \end{aligned} \quad (\text{B6})$$

Thus, we get

$$\sum_i \langle \psi' | \frac{\hat{n}_i}{\hat{N}} |\psi'\rangle \langle \psi' | \frac{\hat{n}_i}{\hat{N}} |\psi'\rangle + d \sum_i |\langle \psi' | \frac{a_i^\dagger a_{i+1}}{\hat{N}} |\psi'\rangle|^2 + \sum_{i \neq j} \omega^{i-j} \langle \psi' | \frac{\hat{n}_i}{\hat{N}} |\psi'\rangle \langle \psi' | \frac{\hat{n}_j}{\hat{N}} |\psi'\rangle. \quad (\text{B7})$$

The upper bound of middle term is

$$d \sum_i \left\| \langle \psi' | \frac{a_i^\dagger}{\sqrt{\hat{N}}} \right\|^2 \left\| \frac{a_{i+1}}{\sqrt{\hat{N}}} |\psi'\rangle \right\|^2 = d \sum_i \langle \psi' | \frac{\hat{n}_i}{\hat{N}} |\psi'\rangle \langle \psi' | \frac{\hat{n}_{i+1}}{\hat{N}} |\psi'\rangle, \quad (\text{B8})$$

while the last term is, due to the fact that for $d = 3$ and for $i \neq j$ one has $\omega + \omega^{-1} = -1$, given by

$$- \sum_i \langle \psi' | \frac{\hat{n}_i}{\hat{N}} |\psi'\rangle \langle \psi' | \frac{\hat{n}_{i+1}}{\hat{N}} |\psi'\rangle. \quad (\text{B9})$$

Finally, we see that the upper bound of $\sum_{m=0}^3 \langle \hat{R}_m \rangle \langle \hat{R}_m^\dagger \rangle$ is $\max \left(\sum_i \langle \psi' | \frac{\hat{n}_i}{\hat{N}} |\psi'\rangle \right)^2 = 1$.

Appendix C: Losses induced noise ($d = 3$)

In this Appendix we want to study noise robustness of the criteria. We shall present here the results for the $d = 3$ case. As our model of noise we shall take losses of photon counts due to inefficiency of the detection. We shall assume that all detectors have the same efficiency.

In the case of a theoretical description of detection/collection losses, an inefficient detector can be emulated by a perfect one with a beam-splitter of transmittivity η in front of it, so that the probability that m photons are counted while n reach beam-splitter reads

$$q(m, n, \eta) = \begin{cases} \delta_{m,0} & \eta = 0 \\ \delta_{m,n} & \eta = 1 \\ \binom{n}{m} \eta^m (1 - \eta)^{n-m} & \text{otherwise} \end{cases} \quad (\text{C1})$$

We consider the six-mode bright squeezed vacuum state, which up to local transformations reads

$$\begin{aligned}
|\Psi\rangle &\propto \sum_{i=1}^{10 \text{ (numerical cutoff)}} \tanh^i \Gamma \sqrt{\frac{(i+1)(i+2)}{2}} |\psi^i\rangle, \\
|\psi^i\rangle &= \sqrt{\frac{2}{(i+1)(i+2)}} \sum_{i_1=0}^i \sum_{i_2=0}^{i-i_1} |i_1, i_2, i-i_1-i_2, i_1, i_2, i-i_1-i_2\rangle.
\end{aligned} \tag{C2}$$

In inequality (13), the correlation function for the $2i$ components $|\psi^i\rangle$ for one possible choice of mutually unbiased bases at each side reads ($\omega = \exp(2\pi i/3)$) :

$$\begin{aligned}
&\left\langle \left| \hat{R}_{m,A} - \hat{R}_{m,B} \right|^2 \right\rangle_i \\
&= \frac{2}{(i+1)(i+2)} \sum_{i_1=0}^i \sum_{i_2=0}^{i-i_1} \sum_{m_{1,A}=0}^{i_1} \sum_{m_{2,A}=0}^{i_2} \sum_{m_{3,A}=0}^{i-i_1-i_2} \sum_{m_{1,B}=0}^{i_1} \sum_{m_{2,B}=0}^{i_2} \sum_{m_{3,B}=0}^{i-i_1-i_2} \\
&\quad q(m_{1,A}, i_1, \eta) q(m_{2,A}, i_2, \eta) q(m_{3,A}, i-i_1-i_2, \eta) q(m_{1,B}, i_1, \eta) q(m_{2,B}, i_2, \eta) q(m_{3,B}, i-i_1-i_2, \eta) \\
&\quad \times \left| \left(\frac{m_{1,A} + \omega m_{2,A} + \omega^2 m_{3,A}}{m_{1,A} + m_{2,A} + m_{3,A}} \right)' - \left(\frac{m_{1,B} + \omega m_{2,B} + \omega^2 m_{3,B}}{m_{1,B} + m_{2,B} + m_{3,B}} \right)' \right|^2
\end{aligned} \tag{C3}$$

(with $(\cdot)'$ meaning that the fraction takes value 0 when no photons were detected at a particular side), while in bound for separable state we will use

$$\begin{aligned}
&\left\langle \Pi_X \frac{1}{\hat{N}_X} \Pi_X \right\rangle_i \\
&= \frac{2}{(i+1)(i+2)} \sum_{i_1=0}^i \sum_{i_2=0}^{i-i_1} \sum_{m_{1,A}=0}^{i_1} \sum_{m_{2,A}=0}^{i_2} \sum_{m_{3,A}=0}^{i-i_1-i_2} \\
&\quad q(m_{1,A}, i_1, \eta) q(m_{2,A}, i_2, \eta) q(m_{3,A}, i-i_1-i_2, \eta) \left(\frac{1}{m_{1,X} + m_{2,X} + m_{3,X}} \right)'.
\end{aligned} \tag{C4}$$

Because of the symmetry mutually for conjugated unbiased bases at both sides, the correlation function remains the same for all pairs of bases. In the next step, these averages are summed with weights over different numbers of photons,

$$\begin{aligned}
\left\langle \left| \hat{R}_{m,A} - \hat{R}_{m,B} \right|^2 \right\rangle_\Gamma &= \frac{1}{C} \sum_{i=1}^{10} \frac{2}{(i+1)(i+2)} \tanh^{2i} \Gamma \left\langle \left| \hat{R}_{m,A} - \hat{R}_{m,B} \right|^2 \right\rangle_i, \\
\left\langle \Pi_X \frac{1}{\hat{N}_X} \Pi_X \right\rangle_\Gamma &= \frac{1}{C} \sum_{i=1}^{10} \frac{2}{(i+1)(i+2)} \tanh^{2i} \Gamma \left\langle \Pi_X \frac{1}{\hat{N}_X} \Pi_X \right\rangle_i, \\
C &= \sum_{i=1}^{10} \frac{2}{(i+1)(i+2)} \tanh^{2i} \Gamma.
\end{aligned} \tag{C5}$$

Plugging this into inequality (13) we get that if

$$4 \left\langle \left| \hat{R}_{m,A} - \hat{R}_{m,B} \right|^2 \right\rangle_\Gamma - 3 \left\langle \Pi_A \frac{1}{\hat{N}_A} \Pi_A \right\rangle_\Gamma - 3 \left\langle \Pi_B \frac{1}{\hat{N}_B} \Pi_B \right\rangle_\Gamma < 0, \tag{C6}$$

the state is entangled. For Γ very large we find that the LHS of (C6) is less than 0 for $\eta > 0.153$. For $\Gamma \approx 0$, we get requirement of $\eta_{critical} \approx 1/4$. This the critical value of the detection efficiency for the criterion (14) based on photon numbers rather than rates.

[1] M. Zukowski, A. Zeilinger, and M. A. Horne, Phys. Rev. A **55**, 2564 (1997).

- [2] N. G. Walker and J. E. Carroll, *Opt. Quantum Electron.* **18**, 355 (1986); N. G. Walker, *J. Mod. Opt.* **34**, 15 (1987).
- [3] A. Zeilinger, H. J. Bernstein, D. M. Greenberger, H. A. Horne, M. Żukowski, Controlling Entanglement in Quantum Optics, in *Quantum Control and Measurement*, H. Ezawa, Y. Murayama (Editors), (Elsevier Sci. Publ. B. V. , 1993); A. Zeilinger, M. Żukowski, M. A. Horne, H. J. Bernstein, D. M. Greenberger, Einstein-Podolsky-Rosen correlations in higher dimensions, in *Fundamental Aspects of Quantum Theory*, Eds. F. DeMartini, G. Denardo, A. Zeilinger, (Singapore, World Scientific, 1993).
- [4] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, *Phys. Rev. Lett.* **73**, 58 (1994).
- [5] D. Kaszlikowski, P. Gnaniński, M. Żukowski, W. Miklaszewski, and A. Zeilinger *Phys. Rev. Lett.* **85**, 4418 (2000).
- [6] J.-L. Chen, D. Kaszlikowski, L. C. Kwek, C. H. Oh, and M. Żukowski, *Phys. Rev. A* **64**, 052109 (2001)
- [7] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, *Phys. Rev. Lett.* **88**, 040404 (2002).
- [8] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- [9] J.-W. Pan, Z.-B. Chen, C.-Y. Lu, H. Weinfurter, A. Zeilinger, and M. Żukowski, *Rev. Mod. Phys.* **84**, 777 (2012).
- [10] C. Simon and D. Bouwmeester, *Phys. Rev. Lett.*, **91**, 053601 (2003).
- [11] T. Sh. Iskhakov, I.N. Agafonov, M.V. Chekhova, and G. Leuchs, *Phys. Rev. Lett.* **109**, 150502 (2012); M. Stobińska, F. Töppel, P. Sekatski, and M. V. Chekhova, *Phys. Rev. A* **86**, 022323 (2012).
- [12] M. Żukowski, W. Laskowski, and M. Wiesniak, arXiv:1508.02368.
- [13] C. Schaeff, R. Polster, M. Huber, S. Ramelow, A. Zeilinger, *Optica* **2**, 523 (2015), see also arXiv:1502.06504; for early multiport experiments see K. Mattle, M. Michler, H. Weinfurter, A. Zeilinger, M. Żukowski *Applied Physics B-Lasers and Optics* **60**, S111-S117 (1995).
- [14] M. Żukowski, W. Laskowski, and M. Wiesniak, arXiv:1506.08732.
- [15] W. K. Wootters and B. D. Fields, *Ann. Phys.* **191**, 363 (1989); I. D. Ivanovic, *J. Phys. A* **14**, 3241 (1981).